AVERAGE AND PROBABILITY.

167. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A line l is divided into n segments by $n-1$ points taken at random on it; find the mean value of the product of p of the segments, the p segments being taken at random and p being less than n.

Solution by the PROPOSER.

Let the p segments taken at random in one instance be a, b, c, ... If p other points are taken at random on the line, the chance that the first will fall on a is a/l; that the second will fall on b is b/l; and so on. Hence, the chance that the p points will all fall on a, b, c, ... in a given order is $M(a.b.c...)/l^p$; and the chance that they will so fall in any order is $(1.2.3...p)M(a.b.c...)/l^p$.

The whole number of ways in which the $n+p-1$ points can be arranged is $(n+p-1)!$ The whole number of ways in which p points can fall on p segments (each in any order) is $(p!)(p!)$. We may then have

$$\frac{(1.2.3...p)\,M(a.b.c...)}{l^p}=\frac{(1.2.3...p)^2}{1.2.3...(n+p-1)}.$$

Hence, $M(a.b.c...)=\dfrac{(1.2.3...p)l^p}{1.2.3...(n+p-1)}$.

172. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A circular arc, with center at one corner of a given square, is drawn through a point taken at random in the square. What is the average length of the arc within the square?

Solution by B. F. FINKEL, Ph. D., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let $ABCD$ be the square, whose side is a, and let the coördinates of the random point P be (x, y), A being origin, AB the x axis and AD the y axis. Then the length of the arc through the random point is $2\phi\sqrt{(x^2+y^2)}$, where 2ϕ is the angle between the lines drawn from the origin to the intersection of the arc with the sides of the square. Passing to polar coördinates, we have $2\phi\rho$ for the length of the arc. But $\phi=\frac{1}{4}\pi-\sec^{-1}(\rho/a)$. Hence, the length of the arc is $2\rho[\frac{1}{4}\pi-\sec^{-1}(\rho/a)]=f(\rho)$, (say). Then the average length of the arc is

$$\Delta=\frac{f(\rho_1)+f(\rho_2)+\ldots+f(\rho_n)}{n},\text{ where } n \text{ is the number of arcs,}$$

$$=\frac{\int_0^a 2\rho\frac{\pi}{4}d\rho+2\int_a^{\sqrt{2}a}\rho\left(\frac{\pi}{4}-\sec^{-1}\frac{\rho}{a}\right)d\rho}{\int_0^a d\rho+\int_a^{\sqrt{2}a}d\rho}$$

$$=\left[2\int_0^{\sqrt{2a}}\rho.\frac{\pi}{4}.d\rho-2\int_a^{\sqrt{2a}}\left(\sec^{-1}\frac{\rho}{a}\right)d\rho\right]/\sqrt{2}\,a=\tfrac{1}{2}\sqrt{2a}.$$

Solved in a similar manner with the same result by G. B. M. Zerr.
Solved in an entirely different manner with the result

$$\Delta=4a/3[\sqrt{2}-1-\int_0^{\frac{1}{4}\pi}\log(\tan\tfrac{1}{4}\pi+\tfrac{1}{2}\theta)d\theta]$$

by Henry Heaton. Mr. Heaton assumes that the number of arcs of any given length within the square is proportional to the lengths of the arcs, and the whole number of arcs is equal to the number of points in the square. Now there is no reason why such an assumption may not be made, but such an assumption is certainly highly artificial. In our solution, it is clear that for every point we get a corresponding arc, and but one. If we take in all possible points in the square, we get all possible arcs. How to get all possible points is an open question, and an indefinite number of assumptions may be made as regards the distribution of the points when no law of distribution is given in the problem. The above solution tacitly assumes, (1) That the random points are distributed at equal angular distances on the arcs of circles, and (2) that the arcs of the circles cut the diagonal of the square at equal distances apart. By these assumptions every point in the square is considered. But either or both of these assumptions may be changed in any way at pleasure, each change giving different answers. Ed. F.

PROBLEMS FOR SOLUTION.

ALGEBRA.

273. Proposed by THEODORE L. DELAND, Treasury Department, Washington, D. C.

Three ingots of the precious metals were received at the Mint for assay, where it was found as follows: That in 3 grains of the first ingot and 2 grains of the second the gold was 3 times the silver; that in 2 grains of the first and 6 grains of the third the gold was 8 times the copper; that in 2 grains of the second and 3 grains of the third the silver was 5 times the copper; that in 1 grain of the first, 2 grains of the second, and 3 grains of the third the gold was 2 times the silver; that in 1 grain each of the first and second ingots there were 11 parts of gold to 5 parts of silver; and that 6 grains of the first, 5 grains of the second, and 2 grains of the third on being assayed proved to be 17 carats gold fine. There was no trace of any other metal in the ingots.

Required: The theoretical analysis of each of the three ingots.

274. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find the limit of $\frac{3^2+1}{3^2-1}.\frac{5^2+1}{5^2-1}.\frac{7^2+1}{7^2-1}.\frac{11^2+1}{11^2-1}\ldots$ where the squared numbers are the natural odd *primes* in order.